Andrew, Pehrson
11/8/16

Math synopsis

Contents

1 Numbers, Types of Numbers, & Counting systems

Numbers are arithmetic values expressed by symbols, figures, or words, that are used in counting and calculations. Values can be given many different symbols that organize into counting systems. Counting systems are determined by their position notation. examples of counting systems are the decimal system used predominantly in the world, the binary system used for electronics and data transfer, and the hexadecimal counting systems used on the Chinese abacus, or Suanpan. Numbers are not just organized by there counting system but there number type. Number types are dependent off of the numbers value such as its sine and fractions of a point.

The table below shows a comparison of binary, decimal, and hexadecimal counting systems.

Binary	Decimal	Hexa-Decimal
0	0	0
1	1	1
10	2	2
11	3	3
100	4	4
101	5	5
110	6	6
111	7	7
1000	8	8
1001	9	9
1010	10	A
1011	11	B
1100	12	C
1101	13	D
1110	14	E
1111	15	F
10000	16	10

The highlighted portion of the table above stop at the start of a new order of magnitude

Below the types of numbers are shown in order of inclusion

Natural Numbers	$\mathbb{N}$	$\{1,2,3,...\}$
Whole Numbers	$\mid \mathbb{Z} \mid$	$\{0,1,2,3,...\}$
integers	$\mathbb{Z}$	$\{\mathbb{N}\}\{...,-2,-1,0\}$
Rational Numbers	$\mathbb{Q}$	$\{\mathbb{Z}\}\{..., \frac{1}{-2}, \frac{1}{2}, ...\}$
Irrational Numbers	$\mathbb{I}$	$\{\pi, \sqrt{3}\}$
Real Numbers	$\mathbb{R}$	$\{\mathbb{Q}, \mathbb{I}\}$
Complex Numbers	$\mathbb{C}$	$\{i^2 = -1 \therefore a + bi \in \mathbb{C}\}$

2 Expressions, Equations & Solutions

An expression is any grouping of values and operators without an equality. Expression can be simplified but can't be solved. Equations are two expressions put together as an equality. Equations can be solved. A solution is when an equations is manipulated, due to laws of equality, and a single variable is on one side of the equality. In solving for y in the equation

$$2x^3 - 4y^2 = 8$$

the first thing to do is to subtract both sides by $2x^3$. This is possible due to the subtraction property of equality.

$$(2x^3 - 4y^2) - 2x^3 = (8) - 2x^3$$

Simplifying this we're given

$$-4y^2 = 8 - 2x^3$$

Next -4 is divided out of both sides and this is aloud do to the division property of equality.

$$\frac{(-4y^2)}{-4} = \frac{(8 - 2x^3)}{-4}$$

Simplifying this you get

$$y^2 = \frac{(8 - 2x^3)}{-4}$$

Finally the exponent on y is cancelled out by taking the value of the exponent and putting the expression to the reciprocal power of the exponent. This is then done to the other side of the equation do to the exponential property of equality. In this example the reciprocal of the exponent 2 is $\frac{1}{2}$ which is also written as $\sqrt{}$.

$$\sqrt{y^2} = \sqrt{\frac{(8 - 2x^3)}{-4}}$$

This is then simplified to

$$y = \sqrt{\frac{(8 - 2x^3)}{-4}}$$

Which is the equation solved for y.

3 Functions & domains

Functions are an equations were two variable exist, and the equation is set to salve for one of the variables. This can be seen in the equation...

$$y - 5 = 3(x - 2)$$
$$y = 3(x - 2) - 5$$

This creates a relationship between the two variables where one variable has to be input into the equation to solve for the other variable. To show this relationship the dependent variable, the variable set up to be salved, can be replaced with f(x) . f(x) shows that the value of y is depended on the value of x.

$$y = 3(x-2) - 5$$
$$f(x) = 3(x-2) - 5$$

For salving a function the value of x has to be plugged in.

$$\begin{aligned} x &= 2 \\ f(x) &= 3(x-2) - 5 \\ f(2) &= 3(2-2) - 5 \\ &= 3(0) - 5 \\ &= -5 \\ f(x) &= 5 \end{aligned}$$

The domain of a function are all of the permissible inputs that give a real number. For the equation...

$$f(x) = \frac{2x^2 - 2x + 1}{4 - x^2}$$

The domain can be found by looking for what value of x will give you an answer of $\frac{n}{0}$. Finding what value will do this you can take the denominator and set it equal to 0 and solve for x.

$$\begin{aligned} 4 - x^2 &= 0 \\ (4 - x^2) - 4 &= (0) - 4 \\ -x^2 &= -4 \\ -1(-x^2) &= -1(-4) \\ x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ x &= 2 \end{aligned}$$

The domain for the function...

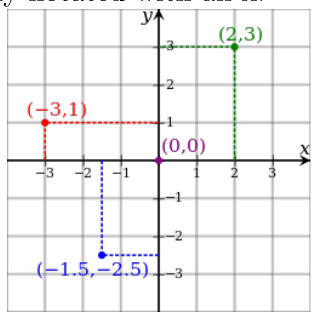
$$f(x) = \frac{2x^2 - 2x + 1}{4 - x^2}$$

is

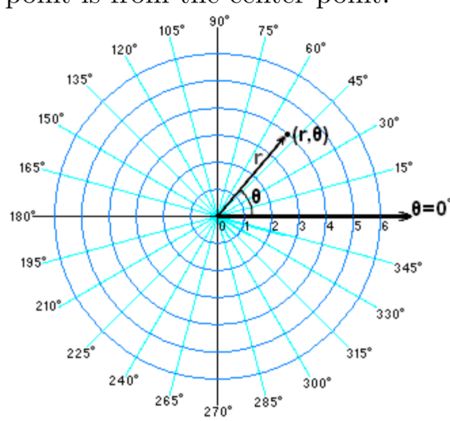
$$\mathbb{R} \neq 2$$

4 Coordinate Systems

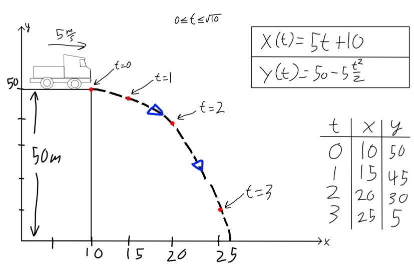
Cartesian systems are defined by a vertical axises and a horizontal axises in witch the vertical is the dependent variable axises normally notated with a y and the horizontal is the independent variable normally notated with an x.



polar systems are defined buy a center point and a radius. the independent variable in the system is the angle the the point is from the center point in radians. the depended variable is the distance the point is from the center point.



Parametric equations have a third parameter t. This parameter t is shown as a progression along the function with a given position from the x, and y parameters. In the drawing below t represents time in seconds while x represents movement in the horizontal and y represents movement in the vertical. A parametric equation was written using kinematic equations on projectile motion. With this it can be seen that on this function there is a direction of time progression.



parametric systems work almost exactly like Cartesian systems, but unlike Cartesian both the vertical and horizontal axises are the depended variable and the parameter t is the independent variable. parametric equations are formatted as...

$$X(T) = b_1T - a_1$$
$$Y(T) = b_2T - a_2$$

Where X is the horizontal depended variable
Y is the vertical depended variable
a is a constant
b is a constant
T is the the independent variable

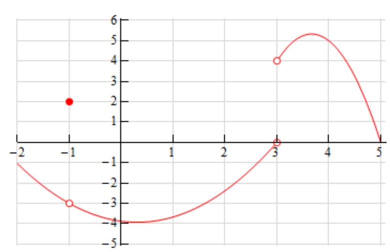
5 Limits

In creating a graph off of the following limits

$\lim_{x \to 3^-} f(x) = 0 \quad \lim_{x \to 3^+} f(x) = 4 \quad$ f(3) does not exist

$\lim_{x \to -1} f(x) = -3 \qquad f(-1) = 2$

First the points f(-1) and f(3) should be explained graphically. f(-1) equals 2, this means that a solid point will be at the coordinates (-1,2). The limit as x approaches -1 from both sides is equal to -3, this means that there will be a hollow dot at coordinates (-1,-3) with a line intersecting it. f(3) does not exist meaning there is no point at 3 on the x axis. The limit as x approaches 3 from the negative side is 0 and as it approaches from the positive side the limit is 4, this means that a hollow dot at the coordinates (3,0) will have a line extending out towards lower x values and a hollow dot at (3,4) and a line extending towards higher x values.



for evaluating the limit:

$$\lim_{x \to -1} \left(\frac{x^2 - 2x - 3}{x + 1} \right)$$

First the limit is evaluated by substituting x for the infinitesimal point being approached:

$$\lim_{x \to -1} \left(\frac{x^2 - 2x - 3}{x + 1} \right)$$

$$\lim_{x \to -1} f(x) = \left(\frac{(-1)^2 - 2(-1) - 3}{(-1) + 1} \right)$$

$$\lim_{x \to -1} f(x) = \left(\frac{1 + 2 - 3}{0} \right)$$

$$\lim_{x \to -1} f(x) = \left(\frac{3 - 3}{0} \right)$$

$$\lim_{x \to -1} f(x) = \left(\frac{0}{0} \right)$$

Since the limit evaluates to $\frac{0}{0}$ the function has to be simplified by factoring the numerator

$$\lim_{x \to -1} \left(\frac{x^2 - 2x - 3}{x + 1} \right)$$

$$\lim_{x \to -1} \left(\frac{(x - 3)(x + 1)}{x + 1} \right)$$

The function factors to this because of a mental shortcut were the numbers in the factored expression are the terms to the summation of the value being multiplied by a singular x and the terms to the multiplication of the singular value. Next the (x+1)'s cancel out

$$\lim_{x \to -1} \left(\frac{(x - 3)(x + 1)}{x + 1} \right)$$

$$\lim_{x \to -1} \left(\frac{(x - 3)1}{1} \right)$$

$$\lim_{x \to -1} (x - 3)$$

finally x is replaced by the infinitesimal point being approached

$$\lim_{x \to -1} (x - 3)$$

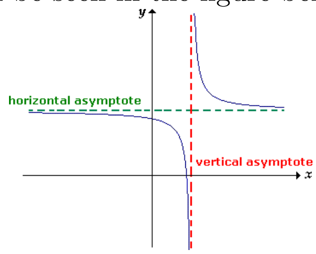
$$\lim_{x \to -1} f(x) = ((-1) - 3)$$

$$\lim_{x \to -1} f(x) = (-4)$$

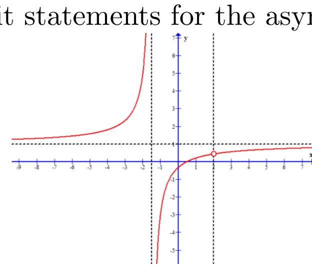
Now that the limit is evaluated it is known that the line approaches -1 at -4 on the y axes.

6 Limits & vertical/horizontal asymptotic behavior

asymptotic behavior is defined as a curve that infinitely approaches a point wile never reaching it. the most common kinds of asymptotes that occur in functions are vertical and horizontal asymptotes. these can be seen in the figure below.



below a graph of a curve with both horizontal and vertical asymptotes is shown. under the figure the corresponding limit statements for the asymptotes.



$$\lim_{x \to -1\frac{1}{2}^-} f(x) = \infty$$

$$\lim_{x \to -1\frac{1}{2}^+} f(x) = -\infty$$

$$\lim_{x \to -\infty} f(x) = 4$$

$$\lim_{x \to \infty} f(x) = 4$$

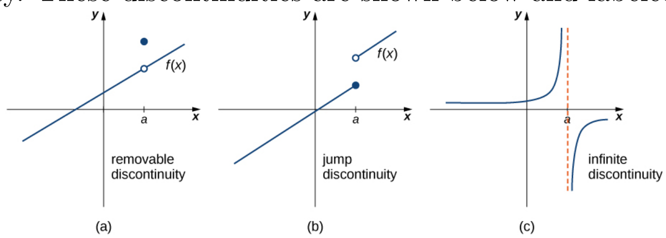
This explains how as x approaches $-1\frac{1}{2}$ from the negative side f(x) approaches positive infinity, and from the positive side f(x) approaches negative infinity. This also explains how as x approaches negative infinity (going off to the left of the graph) f(x) approaches 4, and as x approaches positive infinity (going off to the right of the graph) f(x) approaches 4.

7 Continuity

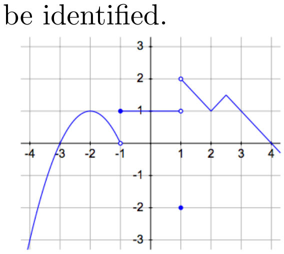
Continuity is a segment on a function where no point is missing. a segment is continuous if...

$$f(x) = \text{exists}$$
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

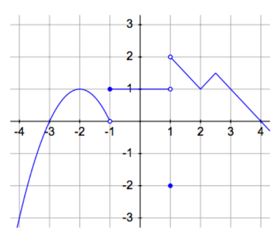
Discontinuity can be seen graphically as a removable discontinuity, step discontinuity, and infinite discontinuity. These discontinuities are shown below and labeled.



These examples can be used to get an understanding of where a discontinuity is on a graph and how it behaves. With the behaviour, limits can be written. For the graph below these types of discontinuities can be identified.



At x=-1 the function has a step discontinuity. At x=1 the function has a step discontinuity and a removable discontinuity. With this knowledge of the discontinuities limits could be written.



For the graph above the discontinuities can be explained through limits.

$$\lim_{x\to -1^-} f(x) = 0$$
$$\lim_{x\to -1^+} f(x) = 1$$
$$\lim_{x\to 1^-} f(x) = 1$$
$$\lim_{x\to 1^+} f(x) = 2$$
$$f(-1) = 1$$
$$f(1) = -2$$

8 Slopes & rates of change (average versus instantaneous)

average rate of change is defined buy the equation...

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

average rate of change describes the slope of the secant line to the curve. instantaneous rate of change is defined buy the equation...

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

the instantaneous rate of change explains the slope of the tangent line to a curve.

9 Derivatives

Derivatives can be described by the equation...

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

this equation is the slope formula, $\frac{y_2-y_1}{x_2-x_1}$, with a limit statement. this derivative definition can be used to find the instantaneous slope of a single point "a". functions such as...

$$f(x) = x^2 - 2x + 1$$

can be plugged into the slope definition to find the instantaneous slope of point "a" lets say 4

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$
$$\lim_{x\to 4} \frac{(x^2-2x+1)-((4)^2-2(4)+1)}{x-4}$$
$$\lim_{x\to 4} \frac{(x^2-2x+1)-(9)}{x-4}$$
$$\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$$

now the function is simplified algebraically to get x-4 out of the denominator, for x will be substituted it 4 and it will give an answer of 0. this is done by factoring.

$$\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$$
$$\lim_{x\to 4} \frac{(x-4)(x+2)}{x-4}$$
$$\lim_{x\to 4} x+2$$

finally the limit statement is salved by substituting x with 4.

$$\lim_{x\to 4} x+2 = 4+2$$
$$= 6$$

now it is known that the derivative of 4 in the function...

$$f(x) = x^2 - 2x + 1$$

is 6.

Derivatives can also be explained by the equation...

$$f^{'}(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

the definition is called the difference quotient. unlike the slope definition the difference quotient finds the equation that describes the derivatives of all points on the original function. functions such as...

$$f(x) = x^2 - 2x + 1$$

can be plugged in to find the equation that explains the derivative of the function

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{((x+h)^2-2(x+h)+1)-(x^2-2x+1)}{h}\right)$$

now the function is simplified algebraically to get h out of the denominator, for h will later be replaced with 0

$$\lim_{h\to 0}\left(\frac{((x+h)^2-2(x+h)+1)-(x^2-2x+1)}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{(x^2+xh+xh+h^2-2x-2h+1)-(x^2-2x+1)}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{(x^2+xh+xh+h^2-2x-2h+1)-x^2+2x-1}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{x^2+x^2+xh+xh+h^2+2x-2x-2h+1-1}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{2xh+h^2-2h}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{h(2x+h-2)}{h}\right)$$
$$\lim_{h\to 0} 2x+h-2$$

finally the limit statement is salved by substituting h for 0.

$$\lim_{h\to 0}(2x+h-2) = 2x+0-2$$
$$= 2x-2$$

now it is known that the derivative of...

$$f(x) = x^2-2x+1$$

is...

$$f^{'}(x) = 2x-2$$

the compare this definition to the previous one the point at x=4 can be plugged in

$$\begin{aligned} f^{'}(x) &= 2x - 2 \\ f^{'}(4) &= 2(4) - 2 \\ &= 8 - 2 \\ &= 6 \end{aligned}$$

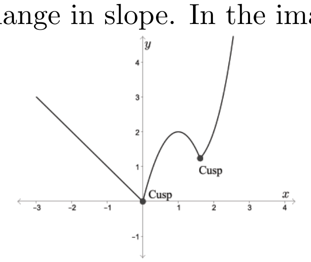
it can be seen that both definitions give the same answer.

10 Differentiability

Differentiability is were continuity exists for a graph of the derivative. a segment is differentiable if...

$$\begin{aligned} f(x) &= \text{exists and is continuous} \\ \lim_{x \to a^-} \frac{f(x+h) - f(x)}{h} &= \lim_{x \to a^+} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

On a function where ever it is continuous it is differentiable except for at cusps. cusps are a point were there is a sudden change in slope. In the image below the cusps are labeled.



11 L'Hopital's rule

some times when salving limit statements for a derivative the equation cannot be manipulated to salve for anything other then $\frac{0}{0}$. this is were L'Hopital's rule comes into play. L'Hopital's rule states that if a limit statement is indeterminate, to salve for the limit the numerator is differentiated and the denominator is differentiated. the limit is then taken of the new equation. for L'Hopital's rule to work the function the limit is being taken of has to be in a rational form. as long as the limit is indeterminate and in a rational form L'Hopital's rule applies.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)'}{g(x)'}$$

for the limit below L'Hopital's rule can be used to salve it.

$$\lim_{x \to 0} \frac{x^2 + 2x}{2x^2}$$

before using L'Hopital's rule the function has to be proven to be indeterminate.

$$f(x) = \frac{x^2 + 2x}{2x^2}$$
$$\lim_{x \to 0} f(x) = \frac{(0)^2 + 2(0)}{2(0)^2}$$
$$= \frac{0}{0}$$

using L'Hopital's rule once gives you the function...

$$\lim_{x \to 0} \frac{2x + 2}{4x + 0}$$

when the limit is evaluated it is still $\frac{0}{0}$.

$$f(x) = \frac{2x + 2}{4x}$$
$$\lim_{x \to 0} f(x) = \frac{2(0) + 2}{4(0)}$$
$$= \frac{0}{0}$$

using L'Hopital's rule once gives you the function...

$$\lim_{x \to 0} \frac{2 + 0}{4}$$

when the limit is evaluated it is $\frac{1}{2}$.

$$f(x) = \frac{2}{4}$$
$$\lim_{x \to 0} f(x) = \frac{2}{4}$$
$$= \frac{1}{2}$$

12 Derivatives rules: description (power, product, quotient, chain)

Derivative rules are a short hand to finding the derivative of a function. the power rule is used when finding the derivative of a function consisting of an exponent. the product rule is used when finding the derivative of the product of two or more functions. the quotient rule is used when finding the derivative of a quotient of two functions. finally the chain rule is used when finding the derivative of a composite function. these rules are all most always used to find the derivative, replacing the derivative definition.

- **Derivatives - example(s) for power rule**

  the power rule is used when a function is put to the power of some value n.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

  for finding the derivative of the function below the power rule is used.

$$\frac{d}{dx}(x^4 + 2x^3 + 12x)$$

  to make this more clear to understand $\frac{d}{dx}$ can be written next to every function separated by an addition operator.

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(2x^3) + \frac{d}{dx}(12x)$$

  now it is easier to see that the derivative of each function in between the addition operator is salved.

$$\begin{aligned}\frac{d}{dx}(x^4) &= 4x^{4-1}\\ &= 4x^3\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(2x^3) &= 2 \cdot 3x^{3-1}\\ &= 6x^2\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(12x) &= 12 \cdot 1x^{1-1}\\ &= 12x^0\\ &= 12\end{aligned}$$

  this gives a final derivative of

$$4x^3 + 6x^2 + 12x + 12$$

- **Derivatives - example(s) for product rule**

  the product rule is used when finding the derivative of the product of two or more functions.

$$\frac{d}{dx}(uv) = u'v + v'u$$

  for finding the derivative of the function below the product rule is used.

$$\frac{d}{dx}(2x \cdot x^2)$$

  this function can be plugged into the product rule.

$$\frac{d}{dx}(2x \cdot x^2) = \frac{d}{dx}(2x) \cdot x^2 + \frac{d}{dx}(x^2) \cdot 2x$$

  the derivatives of the expressions can now be found and the function simplified.

$$\begin{aligned} \frac{d}{dx}(2x) \cdot x^2 + \frac{d}{dx}(x^2) \cdot 2x \\ 2 \cdot x^2 + 2x \cdot 2x \\ 2x^2 + 2x^2 \\ 4x^2 \end{aligned}$$

  this shows that the derivative of $2x \cdot x^2$ is $4x^2$.

- **Derivatives - example(s) for quotient rule**

  the quotient rule is used when finding the derivative of a quotient of two functions.

$$\frac{d}{dx}(\frac{u}{v}) = \frac{u'v - v'u}{v^2}$$

  for finding the derivative of the function below the quotient rule is used.

$$\frac{d}{dx}(\frac{x^2}{3x})$$

  the function can be plugged into the quotient rule

$$\frac{d}{dx}(\frac{x^2}{3x}) = \frac{\frac{d}{dx}(x^2) \cdot 3x - \frac{d}{dx}(3x) \cdot x^2}{(3x)^2}$$

the derivatives of the expressions can now be found and the function simplified.

$$\frac{\frac{d}{dx}(x^2)\cdot 3x - \frac{d}{dx}(3x)\cdot x^2}{(3x)^2}$$
$$\frac{2x\cdot 3x - 3\cdot x^2}{3x^2}$$
$$\frac{6x^2 - 3x^2}{3x^2}$$
$$\frac{3x^2}{3x^2}$$
$$1$$

this shows that the derivative to the function $\frac{x^2}{3x}$ is 1.

- **Derivatives - example(s) for chain rule**

  the chain rule is used when finding the derivative of a composite function.

  $$\frac{d}{dx}(out \circ in) = in' \cdot out' \circ in$$

  for finding the derivative of the function below the chain rule is used.

  $$\frac{d}{dx}(sin^{x^2})$$

  this function can be plugged into the chain rule.

  $$\frac{d}{dx}(sin^{x^2}) = \frac{d}{dx}(x^2)\cdot\frac{d}{dx}(\sin x)\circ x^2$$

  the derivatives of the expressions can now be found and the function simplified.

  $$\frac{d}{dx}(x^2)\cdot\frac{d}{dx}(\sin x)\circ x^2$$
  $$2x\cdot\cos xx^2$$
  $$2x\cos(x^2)$$

  this shows that the derivative to the function sin^{x^2} is $2x\cos(x^2)$

13 Derivatives of implicit functions

implicit functions are functions in witch the dependent variable is not isolated to one side of the equations. because of this the shorthand of automatically dividing both sides by dx dose not apply. for finding the derivative of an implicit function it is essential to, after taking the derivative of an expression, denote what variable the derivative was taken of by multiplying the derivative by dx were x is replaced with the appropriate variable. for finding the derivative of the implicit function...

$$y^2 + xy + x^2 = 1$$

first the derivative is separated buy the addition and subtraction operators.

$$d(y^2) + d(xy) + d(x^2) = d(1)$$

next the derivatives are found wile leaving behind the denotation of the variable the derivative was taken of.

$$\begin{aligned} d(1) &= d(y^2) + d(xy) + d(x^2) \\ 0 &= 2ydy + xdy + ydx + 2xdx \end{aligned}$$

now the derivative is manipulated so that everything multiplied by dx is on one side and things multiplied by dy to the other.

$$\begin{aligned} 0 &= 2ydy + xdy + ydx + 2xdx \\ -2ydy - xdy &= ydx + 2xdx \end{aligned}$$

finally dx and dy are factored out and the function is manipulated so that $\frac{dy}{dx}$ is on one side.

$$\begin{aligned} -2ydy - xdy &= ydx + 2xdx \\ dy(-2y - x) &= (y + 2x)dx \\ \frac{dy}{dx} &= \frac{y + 2x}{-2y - x} \end{aligned}$$

this shows that the derivative of the function $y^2 + xy + x^2 = 1$ is $\frac{y+2x}{-2y-x}$

14 Derivatives of parametric functions

for finding the derivative of parametric equations the same thing is done as in implicit functions except to get $\frac{dy}{dx}$ the two separate functions have to be put into a rational form. for finding the derivative of the parametric function...

$$\begin{aligned} x &= 2\cos\theta + \cos(2\theta) \\ y &= 2\sin\theta - \sin(2\theta) \end{aligned}$$

first the derivative of each function is taken separately with respect to the independent variable in both. in this case θ.

$$\frac{d}{d\theta}(x) = \frac{d}{d\theta}(2\cos\theta + \cos(2\theta))$$
$$\frac{dx}{d\theta} = -2\sin\theta - 2\sin(2\theta)$$

$$\frac{d}{d\theta}(y) = \frac{d}{d\theta}(2\sin\theta - \sin(2\theta))$$
$$\frac{dy}{d\theta} = 2\cos\theta + 2\cos(2\theta)$$

Finally the derivative of the function needed to be found is dy with respect to dx; This means that the two derivatives are put into on rational function with dy being on the top. doing this also cancels out $d\theta$.

$$\frac{dy}{dx} = \frac{2\cos\theta + 2\cos(2\theta)}{-2\sin\theta - 2\sin(2\theta)}$$
$$= \frac{\cos\theta + \cos(2\theta)}{-\sin\theta - \sin(2\theta)}$$
$$= \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}$$

finally giving the derivative...

$$\frac{dy}{dx} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}$$

15 Anti-derivatives

Anti-derivatives are the basic and intuitive form of an integral. Anti-derivatives are were the original function, F, are found from the derivative. this is normally done buy a set of rules as listed below.

F	f	f'
$e^x + C$	e^x	e^x
$\frac{1}{n+1} \cdot x^{n+1} + C$	x^n	$n \cdot x^{n-1}$
$-\cos x + C$	$\sin x$	$\cos x$
$\sin x + C$	$\cos x$	$-\sin x$
$\frac{1}{\ln a} \cdot a^x + C$	a^x	$\ln a \cdot a^x$

when taking the anti-derivative of a function it is essential to end the function with +C since unless more information is given it's unknown were on the graph the function lies.

16 Differential Equations & Slope Fields

differential equations are any equation that contains a derivative within it. the steps to salving a differential equation are the revers of those to salving the derivative of an implicit function. for salving the differential equation...

$$\frac{dy}{dx} = 3xy^3$$

first $\frac{dy}{dx}$ is distributed to both sides were all expressions relating to y are on one side and all expressions relating to x are on the other.

$$\frac{dy}{dx} = 3xy^3$$
$$\frac{1}{y^3}dy = 3xdx$$

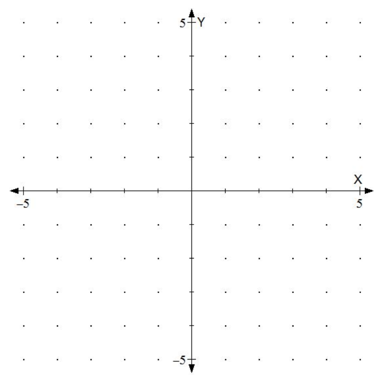
now anti-derivatives are used to revers the derivatives on both sides.

$$\frac{1}{y^3}dy = 3xdx$$
$$\frac{1}{-2}y^-2 + c = \frac{3}{2}x^2 + c$$

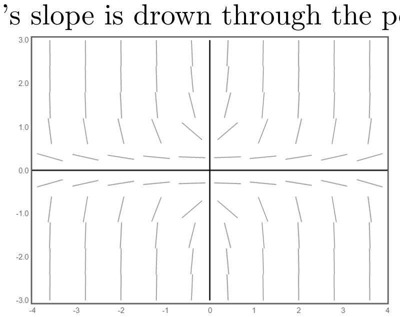
now the equation is salved for y.

$$\frac{1}{-2}y^-2 + c = \frac{3}{2}x^2 + c$$
$$y^-2 = (\frac{3}{2}x^2 + c) - 2$$
$$y^-2 = -3x^2 + c$$
$$y = \frac{1}{\sqrt{-3x^2 + c}}$$

this is the salved form of the differential equation. for graphing the slope Field of the differential equal first the graph needs to be made.



for each point on this graph the coordinates are plugged into the original differential equation and a line with that's slope is drown through the point.



17 Use description - tangent & normal lines

tangent lines are lines that touch a curve at only one point. normal lines are lines that are perpendicular(normal) to the tangent line. in the seance of calculus the tangent line is defined as the slope of a single point on the curve. for finding the normal line of $x^2+y^2=25$ at $x=3$ first the derivative of the function is found.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$$
$$2xdx+2ydy=0$$
$$2xdx=-2ydy$$
$$dx(2x)=dy(2y)$$
$$\frac{dy}{dx}=\frac{2x}{2y}$$
$$\frac{dy}{dx}=\frac{x}{y}$$

After the derivative is found the x component can be plugged into the original function to salve for y.

$$\begin{aligned}(3)^2 + y^2 &= 25\\ 9 + y^2 &= 25\\ y^2 &= 16\\ y &= \{-4, 4\}\end{aligned}$$

Now both the x and y coordinates can be plugged into the derivative to get the slope of the tangent line. to change this to the slope of the normal line, m is changed to $\frac{1}{-m}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{y}\\ \frac{dy}{dx} &= \frac{3}{4}\\ m_{\perp} = \frac{4}{-3}\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{y}\\ \frac{dy}{dx} &= \frac{3}{-4}\\ m_{\perp} = \frac{4}{3}\end{aligned}$$

finally both the slope of the normal line and the coordinates at the point are plugged into the point slope formula of a line.

$$(y - 4) = \frac{4}{-3}(x - 3)$$

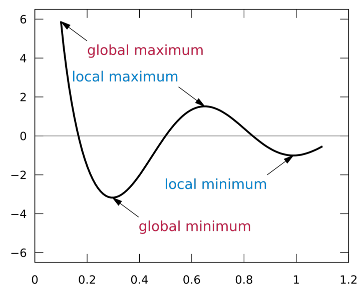
and

$$(y + 4) = \frac{4}{3}(x - 3)$$

18 Optimization and shapes of curves

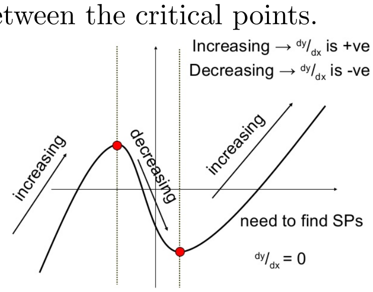
- **Use description - maxima/minima**

  the maxima and minima of a function are collectively known as the extrema. the extrema are the largest and smallest values of the function either within a given interval or over the entire function. the smallest values are called the minima and the largest values are called the maxima. the maxima and minima of an interval is the local maxima and minima. to find the minima and maxima of a function the derivative is set equal to 0 and the critical points are found. these critical points are then plugged into the original function to find the y value and determine whether it is a minima or maxima. when testing the critical points the endpoints of the functions interval are plugged in to test for local maxima and minima.



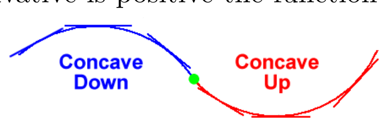
- **Use description - increase/decrease**

a function is increasing at every point on the graph were the slope is positive. the function is decreasing at every point on the graph were the slope is negative. to find whether the function is increasing or decreasing at a point the critical points of the function need to be found and then points from both sides are tested in the derivative function. if the tested point is negative the slope is decreasing for that entire segment in between the critical points. if the tested point is positive the slope is increasing for that entire segment in between the critical points.



- **Use description - concavity**

concavity of the directionality of the curve. concave up is were the function curves upward making a boil. concave down is were the function curves downward making a hill. the concavity of a curve is found by setting the second derivative equal to 0. this finds the inflection points. now points to either side of the inflection are tested in the second derivative function. if the second derivative is negative the curve is concave down. if the second derivative is positive the function is concave up.



- **Optimization**

optimization is identifying the extrema of a function. this is done buy setting the derivative equal to 0 and salving for x and y. optimization is later utilized in related rates and applied to real world problems. for optimizing the function...

$$y = 2xe^x, x \in [-5, 5]$$

first the derivative if the function is found.

$$\begin{aligned} \frac{d}{dx}&(y = 2xe^x) \\ \frac{dy}{dx} &= 2e^x + 2xe^x \end{aligned}$$

after this the function can be set equal to zero and salved for x.

$$\begin{aligned} 2e^x + 2xe^x &= 0 \\ 2xe^x &= -2e^x \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

this shows that at x=-1 an extrema exist on the function $f(x) = 2xe^x$. to find whether it is a minimum or a maximum the value is plugged into the original function to salve for the y along with the end points of the interval.

$$\begin{aligned} y &= 2(-1)e^{(-1)} \\ y &= -2\frac{1}{e} \end{aligned}$$

$$\begin{aligned} y &= 2(-5)e^{(-5)} \\ y &= -10\frac{1}{e^5} \end{aligned}$$

$$\begin{aligned} y &= 2(5)e^{(5)} \\ y &= 10e^5 \end{aligned}$$

this shows that at x=-1 there is a minima and at x=5 a local maxima. to find were the function is increasing and decreasing points from either side of the critical point are plugged into the derivative of the function.

$$
\begin{aligned}
2e^{(-2)} + 2(-2)e^{(-2)} \\
2\frac{1}{e^2} + -4\frac{1}{e^2} \\
-2\frac{1}{e^2}
\end{aligned}
$$

$$
\begin{aligned}
2e^{(0)} + 2(0)e^{(0)} \\
2(0) + 0(1) \\
2
\end{aligned}
$$

this shows that were $x < -1$ the function is decreasing and were $x > -1$ the function is increasing. for finding the concavity of the function the second derivative needs to be taken.

$$
\begin{aligned}
\frac{d}{dx}(\frac{dy}{dx} &= 2e^x + 2xe^x) \\
\frac{d^2y}{dx^2} &= 4e^x + 2xe^x
\end{aligned}
$$

after this the function can be set equal to zero and salved for x.

$$
\begin{aligned}
4e^x + 2xe^x &= 0 \\
2xe^x &= -4e^x \\
2x &= -4 \\
x &= -2
\end{aligned}
$$

this shows that there is concavity to either side of this inflection point. to find the concavities, points to either side of the function need to be plugged into the second derivative.

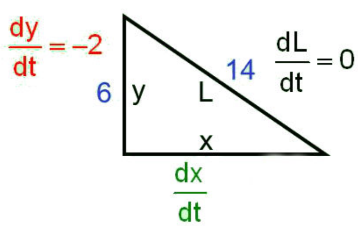
$$
\begin{aligned}
4e^{(-1)} + 2(-1)e^{(-1)} \\
4\frac{1}{e} - 2\frac{1}{e} \\
2\frac{1}{e}
\end{aligned}
$$

$$4e^{(-3)} + 2(-3)e^{(-3)}$$
$$4\frac{1}{e^3} - 6\frac{1}{e^3}$$
$$-2\frac{1}{e^3}$$

this shows that were $x < -2$ the function is concave down and were $x > -2$ the function is concave up.

19 Use description - Related Rates

related rates is using differential equations to find the rate a quantity changes by relating that quantity to other quantity's who's rates of change are known. as an example say a 14 foot ladder is leaning against a wall. if the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the end be moving away form the wall when he top is 6ft above the ground? for this problem whats necessary to take away is that the ladder "slips down the wall at a rate of 2 ft/s" and it has fallen to were the ladder is "6ft above the ground". for this problem $\frac{dx}{dt}$ is the variable being found.



first the length of x needs to be found. this can be done with simple trigonometry.

$$x^2 + y^2 = L^2$$
$$x^2 + 6^2 = 14^2$$
$$x = 4\sqrt{10}$$

next the derivative is taken of the Pythagorean theorem.

$$\frac{d}{dt}(x^2 + y^2 = L^2)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2L\frac{dL}{dt}$$

now the given values can be plugged in and the equation salved for $\frac{dx}{dt}$.

$$
\begin{aligned}
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2L\frac{dL}{dt} \\
2(4\sqrt{10})\frac{dx}{dt} + 2(6)(-2) &= 2(14)(0) \\
\frac{dx}{dt} &= \frac{3}{\sqrt{10}}
\end{aligned}
$$

this gives the answers that the end of the ladder was moving $\frac{3}{\sqrt{10}} ft/s$ away from the wall.